Note on covering and approximation properties (Infinitary combinatorics in set theory and its applications)

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Note on covering and approximation properties

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Abstract

We discuss the covering and approximation properties of an ultrapower of V by a κ-complete ultrafilter over a measurable cardinal κ. Among other things, we prove that it can have both of the μ-covering and μ-approximation properties for every cardinal $\mu > \kappa^+$.

1 Introduction

In this paper we discuss the covering and approximation properties of inner models, which were introduced by Hamkins [3]. First recall these properties: Let M be an inner model, i.e. a transitive inner model of ZFC containing all ordinals, and let μ be a cardinal (in V). Note that $|x| < \mu$ if and only if $|x|^M < \mu$ for any set $x \in M$. We say that M has *the μ-covering property* if for any $x \in [\mathrm{On}]^{<\mu}$ there is $y \in [\mathrm{On}]^{<\mu} \cap M$ with $x \subseteq y$. M is said to have *the μ-approximation property* if a set $A \subseteq \mathrm{On}$ belongs to M whenever $A \cap x \in M$ for all $x \in [\mathrm{On}]^{<\mu} \cap M$.

These properties are often discussed in the context of forcing extensions. It was proved in [3] that if V is a set forcing extension of M by a poset $\mathbb{P}$, and μ is a cardinal with $|\mathbb{P}|^M < \mu$, then M has the μ-covering and μ-approximation properties. Using this fact, Laver [4] proved that a ground model is definable in any set forcing extensions. It was also used in [3] and [4] to prove that certain large cardinals are not created by small forcing extensions. Similar use of these properties can be found in Reiz [5], Fuchs-Hamkins-Reiz [2] and Viale-Weiß [6], too.

In this paper we study the covering and approximation properties of an ultrapower of V by a κ-complete ultrafilter over a measurable cardinal κ. Throughout this paper let κ, U, M and j be as follows:

- κ is a measurable cardinal.
- U is a non-principal κ-complete ultrafilter over κ.
- M is the transitive collapse of ${}^{\kappa}V/U$.
- $j : V \to M$ is the ultrapower map.

Moreover, for each $f \in {}^{\kappa}V$, let $(f)_U \in {}^{\kappa}V/U$ be the equivalence class represented by f, and let $[f]_U \in M$ be the target of $(f)_U$ by the transitive collapse of ${}^{\kappa}V/U$.

Here we summarize our results in this paper. First we present those on the convering property. Note that M has the μ-covering property for every cardinal $\mu \leq \kappa^+$ because ${}^{\kappa}M \subseteq M$. We will obtain the following:

- Assume GCH. Then M has the μ-covering property for every cardinal μ. (Corollary 2.2)
- Assume that ν is a cardinal with $\nu^{<\kappa} = \nu$ and $\nu^{\kappa} > \nu^{+}$. Then M does not have the ν^{++}-covering property. (Corollary 2.3)

Next we present our results on the approximation property. Note that if M has the μ-approximation property, then M has the μ'-approximation property for every $\mu' \geq \mu$. Note also that M does not have the κ^{+}-approximation property because $[U]^{\kappa} \subseteq M$, but $U \notin M$. We will obtain the following:

- Assume that μ is a strongly compact cardinal $> \kappa$. Then M has the μ-approximation property. (Corollary 3.3)
- It is consistent (with GCH) that M has the κ^{++}-approximation property. (Corollary 3.3)
- Suppose that ν is a cardinal $> \kappa$ and that $\square_{\nu}$ holds. Then M does not have the ν^{+}-approximation property. (Corollary 3.8)

Among other things, note that M can have both of the μ-covering and μ-approximation properties for all cardinals $\mu > \kappa^{+}$.

At the end of the introduction we give our notation which may not be standard: For a set A of ordinals, o.t.(A) denotes the order-type of A, and $\mathrm{Lim}(A)$ denotes the set of all limit points of A, i.e. the set of all $\alpha \in A$ such that $A \cap \alpha$ is unbounded in α. For a regular cardinal $\mu > \kappa$ let $E^{\mu}_{<\kappa}$, E^{μ}_{κ} and $E^{\mu}_{>\kappa}$ denote the set of all $\alpha < \mu$ with $\mathrm{cf}(\alpha) < \kappa$, $\mathrm{cf}(\alpha) = \kappa$ and $\mathrm{cf}(\alpha) > \kappa$, respectively. For an elementary embedding k between transtive models of ZFC, $\mathrm{crit}(k)$ denotes the critical point of k.

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2 Covering property

In this section we study the covering property of M. We give a characterization of that M has the μ-covering property for a regular μ:

Proposition 2.1. *The following are equivalent for any regular cardinal μ:*

(i) M has the μ-covering property.

(ii) There is no ordinal ν with $\nu^{+} < \mu \leq j(\nu)$.

Proof. Fix a regular cardinal μ.

First we show that (i) implies (ii). We prove the contraposition. Suppose that there is an ordinal ν with $\nu^{+} < \mu \leq j(\nu)$. Because $|j[\nu^{+}]| < \mu$, it suffices to show that if $j[\nu^{+}] \subseteq y \in M$, then $|y| \geq \mu$.

Suppose that $j[\nu^+] \subseteq y \in M$. We may assume that $y \subseteq j(\nu^+)$. First note that $j[\nu^+]$ is unbounded in $j(\nu^+)$. So y is unbounded in $j(\nu^+)$, too. Then $|y| = j(\nu^+) \geq \mu$ in M because $j(\nu^+)$ is regular in M. Then $|y| \geq \mu$ also in V.

Next we prove the converse. Before starting, note that if x is a set of ordinals with $j(|x|) < \mu$, then there is $y \in [\mathrm{On}]^{<\mu} \cap M$ with $x \subseteq y$: For each $\alpha \in x$ take $f_\alpha : \kappa \to \mathrm{On}$ with $[f_\alpha]_U = \alpha$. Let g be the function on κ defined by $g(\xi) = \{f_\alpha(\xi) \mid \alpha \in x\}$, and let $y := [g]_U$. Then $x \subseteq y$ clearly. Moreover $|g(\xi)| \leq |x|$ for all $\xi \in \kappa$, and so $|y| \leq j(|x|) < \mu$ in M. Then $|y| < \mu$ also in V.

We start to prove that (ii) implies (i). Assume (ii). To prove (i), take an arbitrary $x \in [\mathrm{On}]^{<\mu}$. We must find $y \in [\mathrm{On}]^{<\mu} \cap M$ with $x \subseteq y$.

First suppose that $\mathrm{cf}(|x|) > \kappa$. Then $j(|x|) = \sup_{\nu<|x|} j(\nu)$. But $j(\nu) < \mu$ for all $\nu < |x|$ by (ii) and the fact that $\nu^+ \leq |x| < \mu$. Then $j(|x|) < \mu$ by the regularity of μ. So there is $y \in [\mathrm{On}]^{<\mu} \cap M$ with $x \subseteq y$ by the remark above.

Next suppose that $\mathrm{cf}(|x|) \leq \kappa$. Take a partition $\langle x_\eta \mid \eta < \mathrm{cf}(|x|)\rangle$ of x such that $|x_\eta| < |x|$ for all η. For each η, $j(|x_\eta|) < \mu$ by (ii), and so we can take $y_\eta \in [\mathrm{On}]^{<\mu} \cap M$ with $x_\eta \subseteq y_\eta$. Note that $\langle y_\eta \mid \eta < \mathrm{cf}(|x|)\rangle \in M$ because ${}^\kappa M \subseteq M$. Then it is easy to check that $y := \bigcup_{\eta<\mathrm{cf}(|x|)} y_\eta$ is as desired. □

From Proposition 2.1 we obtain the following corollaries:

Corollary 2.2. *Assume* GCH. *Then M has the μ-covering property for every cardinal μ.*

Proof. $j(\nu) = \nu$ for each $\nu < \kappa$, and $j(\nu) < (\nu^\kappa)^+ \leq \nu^{++}$ for each $\nu \geq \kappa$. So (ii) of Proposition 2.1 holds for every regular cardinal μ. Hence M has the μ-covering property for every regular cardinal μ by Proposition 2.1.

Note that this also implies the μ-covering property of M for every singular cardinal μ: Suppose that μ is a singular cardinal and that $x \in [\mathrm{On}]^{<\mu}$. Then we can take a regular $\mu' < \mu$ with $|x| < \mu'$. By the μ'-covering property of M there is $y \in [\mathrm{On}]^{<\mu'} \cap M$ with $x \subseteq y$. Then $y \in [\mathrm{On}]^{<\mu} \cap M$, and $x \subseteq y$. □

Corollary 2.3. *Assume that ν is a cardinal with $\nu^{<\kappa} = \nu$ and $\nu^\kappa > \nu^+$. Then M does not have the ν^{++}-covering property.*

Proof. By Proposition 2.1 it suffices to show that $\nu^{++} \leq j(\nu)$. First take an injection $\pi : {}^{<\kappa}\nu \to \nu$. For each $b \in {}^\kappa\nu$, define $f_b : \kappa \to \nu$ by $f_b(\xi) = \pi(b \restriction \xi)$. Then the set $\{\xi < \kappa \mid f_b(\xi) = f_{b'}(\xi)\}$ is bounded in κ for any distinct $b, b' \in {}^\kappa\nu$, and so the map $b \mapsto [f_b]_U$ is an injection from ${}^\kappa\nu$ to $j(\nu)$. Hence $\nu^{++} \leq \nu^\kappa \leq j(\nu)$. □

3 Approximation property

In this section we study the approximation property of M. Recall that M does not have the κ^+-approximation property. Here we discuss the μ-approximation property for $\mu > \kappa^+$. In Subsection 3.1 we give a characterization of the μ-approximation property of M for a regular μ. In Subsection 3.2 we prove that M has the μ-approximation property if μ is a generic strongly compact cardinal of some kind. In Subsection 3.3 we show that M does not have the μ-approximation property under a square-like principle at μ.

3.1 Characterization of the approximation property of M

Here we give a characterization of the μ-approximation property of M for a regular $\mu > \kappa$.

First we prepare notation. Let X be a $\subseteq$-directed set. A sequence $\langle f_x \mid x \in X \rangle$ is called a *U-coherent sequence* on X if

(i) $f_x : \kappa \to \mathcal{P}(x)$ (so $[f_x]_U \subseteq j(x)$) for each $x \in X$,

(ii) $\{\xi < \kappa \mid f_y(\xi) \cap x = f_x(\xi)\} \in U$ (i.e. $[f_y]_U \cap j(x) = [f_x]_U$) for each $x, y \in X$ with $x \subseteq y$.

Moreover a U-coherent sequence $\langle f_x \mid x \in X \rangle$ is said to be *U-uniformizable* if there is a function $f : \kappa \to \mathcal{P}(\bigcup X)$ (so $[f]_U \subseteq j(\bigcup X)$) such that $\{\xi < \kappa \mid f(\xi) \cap x = f_x(\xi)\} \in U$ (i.e. $[f]_U \cap j(x) = [f_x]_U$) for all $x \in X$.

Here we prove the following.

Lemma 3.1. *Let μ be a regular cardinal $> \kappa$. Then (i) below implies (ii) below:*

(i) M has the μ-approximation property.

(ii) For any $\lambda \geq \mu$ every U-coherent sequence on $[\lambda]^{<\mu}$ is U-uniformizable.

The converse is also true if $j(\mu) = \mu$.

Proof. Before starting the proof, note that if $y \in [j(\lambda)]^{<\mu} \cap M$ for some $\lambda \geq \mu$, then there is $x \in [\lambda]^{<\mu}$ with $y \subseteq j(x)$: Take $g : \kappa \to \mathcal{P}(\lambda)$ with $[g]_U = y$. We may assume that $|g(\xi)| < \mu$ for all $\xi < \kappa$ because $|y| < \mu \leq j(\mu)$ in M. Then it is easy to see that $x := \bigcup_{\xi<\kappa} g(\xi)$ is as desired.

First we prove that (i) implies (ii). Assume (i). To show (ii) let $\vec{f} = \langle f_x \mid x \in [\lambda]^{<\mu} \rangle$ be a U-coherent sequence for some $\lambda \geq \mu$. Let $A := \bigcup\{[f_x]_U \mid x \in [\lambda]^{<\mu}\}$. Note that $A \subseteq j(\lambda)$ and that $A \cap j(x) = [f_x]_U \in M$ for all $x \in [\lambda]^{<\mu}$ by the coherency of $\vec{f}$. Then $A \cap y \in M$ for all $y \in [\mathrm{On}]^{<\mu} \cap M$ by the remark at the beginning. So $A \in M$ by (i). Take $f : \kappa \to \mathcal{P}(\lambda)$ with $A = [f]_U$. Then $[f]_U \cap j(x) = A \cap j(x) = [f_x]_U$ for all $x \in [\lambda]^{<\mu}$, that is, f U-uniformizes $\vec{f}$.

Next we prove the converse assuming that $j(\mu) = \mu$. Assume (ii). To show (i) suppose that A is a set of ordinals and that $A \cap y \in M$ for all $y \in [\mathrm{On}]^{<\mu} \cap M$. We must show that $A \in M$. Take $\lambda \geq \mu$ with $A \subseteq j(\lambda)$. Here note that if $x \in [\lambda]^{<\mu}$, then $j(x) \in M$, and $|j(x)| < j(\mu) = \mu$. Hence $A \cap j(x) \in M$ for all $x \in [\lambda]^{<\mu}$. For each $x \in [\lambda]^{<\mu}$ take $f_x : \kappa \to \mathcal{P}(x)$ with $[f_x]_U = A \cap j(x)$. Then it is easy to see that $\vec{f} = \langle f_x \mid x \in [\lambda]^{<\mu} \rangle$ is U-coherent. By (ii) take $f : \kappa \to \mathcal{P}(\lambda)$ which U-uniformizes $\vec{f}$. Then $[f]_U \cap j(x) = [f_x]_U = A \cap j(x)$ for all $x \in [\lambda]^{<\mu}$. Moreover $\bigcup\{j(x) \mid x \in [\lambda]^{<\mu}\} = j(\lambda) \supseteq A$ by the remark at the beginning. So $A = [f]_U \in M$. $\square$

3.2 Approximation property for generic strongly compact cardinals

Here we show that if μ is a generic strongly compact cardinal in the following sense, then M has the μ-approximation property: We say that μ is *$\leq\kappa$-closed generic strongly compact* if it satisfies the following:

(i) μ is a regular cardinal $> \kappa^+$.

(ii) For any $\lambda \geq \mu$ there is a $\leq\kappa$-closed forcing extension of V in which we have a (μ, λ)-strongly compact embedding $k : V \to N$. Here $k : V \to N$ is called a (μ, λ)-*strongly compact embedding* if

- N is a transitive model of ZFC.
- k is an elementary embedding with $\mathrm{crit}(k) = \mu$.
- $k[\lambda] \subseteq y$ for some $y \in k([\lambda]^{<\mu})$.

Note that if μ is a strongly compact cardinal $> \kappa$, then in V there is a (μ, λ)-strongly compact embedding for every $\lambda \geq \mu$, and so μ is $\leq \kappa$-closed generic strongly compact. Note also that κ^{++} can be $\leq \kappa$-closed generic strongly compact: Suppose that there is an inner model V' such that $(\kappa^{++})^V$ is strongly compact in V' and such that V is an extension of V' by the Lévy collapse $\mathrm{Col}((\kappa^+)^V, <(\kappa^{++})^V)$. Then it follows from the standard argument that κ^{++} is $\leq\kappa$-closed generic strongly compact in V. (See Cummings [1] for example.) Note also that if GCH holds in V', then so does in V.

As we promised above, we prove the following:

Proposition 3.2. *Suppose that μ is a $\leq\kappa$-closed generic strongly compact cardinal. Then M has the μ-approximation property.*

Corollary 3.3.

(1) If μ is a strongly compact cardinal $> \kappa$, then M has the μ-approximation property.

(2) Suppose that there is an inner model V' such that $(\kappa^{++})^V$ is strongly compact in V' and such that V is an extension of V' by the Lévy collapse $\mathrm{Col}((\kappa^+)^V, <(\kappa^{++})^V)$. Then M has the μ-approximation property.

To prove Proposition 3.2, we need the following lemmata:

Lemma 3.4. *Suppose that μ is a $\leq\kappa$-closed generic strongly compact. Then $\alpha^\kappa < \mu$ for all $\alpha < \mu$, and so $j(\mu) = \mu$.*

Proof. For the contradiction assume that $\alpha < \mu$ and $\alpha^\kappa \geq \mu$. In V take an injection $\tau : \mu \to {}^\kappa\alpha$. Let W be a $\leq\kappa$-closed forcing extension of V and $k : V \to N$ be a (μ, μ)-strongly compact embedding in W. Then it is easy to see that $k(\tau)(\mu) \in ({}^\kappa\alpha)^N \setminus ({}^\kappa\alpha)^V$. So $({}^\kappa\alpha)^N \not\subseteq ({}^\kappa\alpha)^V$. But $({}^\kappa\alpha)^N \subseteq ({}^\kappa\alpha)^W$ because $N \subseteq W$, and $({}^\kappa\alpha)^W = ({}^\kappa\alpha)^V$ becauase W is a $\leq\kappa$-closed forcing extension of V. So $({}^\kappa\alpha)^N \subseteq ({}^\kappa\alpha)^V$. This is a contradiction. □

Lemma 3.5. *Let $\vec{f} = \langle f_x \mid x \in [\lambda]^{<\mu} \rangle$ be a U-coherent sequence for some regular $\mu > \kappa^+$ and some $\lambda \geq \mu$. If $\vec{f}$ is U-uniformizable in some $\leq\kappa$-closed forcing extension of V, then so is in V.*

Proof. Assume that $\mathbb{P}$ is a $\leq\kappa$-closed poset and that $\vec{f}$ is U-uniformizable in $V^{\mathbb{P}}$. Let $\dot{f}$ be a $\mathbb{P}$-name for a function U-uniformizing $\vec{f}$. Because $\mathbb{P}$ is $\leq\kappa$-closed, we can take $p \in \mathbb{P}$ and $S \subseteq \kappa$ (in V) such that $p \Vdash$ "$\{\xi < \kappa \mid \dot{f}(\xi) \in V\} = S$".

Claim. $S \in U$.

Proof of Claim. Take a sufficiently large regular cardinal θ. Because κ is inaccesible, we can take $K \in [\mathcal{H}_\theta]^\kappa$ such that $\kappa, \mu, \lambda, U, \mathbb{P}, p, \dot{f}, S \in K \prec \langle \mathcal{H}_\theta, \in \rangle$ and such that ${}^{<\kappa}K \subseteq K$. Let $z := K \cap \lambda \in [\lambda]^{<\mu}$.

By induction on $\xi < \kappa$ we construct a descending sequence $\langle p_\xi \mid \xi < \kappa \rangle$ in $\mathbb{P} \cap K$. Let $p_0 := p$. If ξ is a limit ordinal, then let $p_\xi \in \mathbb{P} \cap K$ be a lower bound of $\{p_\eta \mid \eta < \xi\}$. We can take such p_ξ because $\mathbb{P}$ is $\leq \kappa$-closed, and ${}^{<\kappa}K \subseteq K$. Finally suppose that ξ is a successor ordinal, say $\xi = \eta + 1$, and that p_η has been taken. If $\eta \in S$, then let $p_\xi := p_\eta$. Otherwise, because $p_\eta \Vdash$ "$\dot{f}(\eta) \notin V$", there are $r_0, r_1 \leq p_\eta$ and $\alpha < \lambda$ such that $r_0 \Vdash$ "$\alpha \in \dot{f}(\eta)$" and $r_1 \Vdash$ "$\alpha \notin \dot{f}(\eta)$". By the elementarity of K we can take such r_0, r_1 and α in K. Let $p_\xi := r_1$ if $\alpha \in f_z(\eta)$, and let $p_\xi := r_0$ if $\alpha \notin f_z(\eta)$. Note that $p_\xi \Vdash$ "$\dot{f}(\eta) \cap z \neq f_z(\eta)$".

Now we have constructed $\langle p_\xi \mid \xi < \kappa \rangle$. By the $\leq \kappa$-closure of $\mathbb{P}$ we can take its lower bound p^*. Then p^* forces that $\dot{f}(\xi) \cap z \neq f_z(\xi)$ for all $\xi \in \kappa \setminus S$. Then $\kappa \setminus S \notin U$ because $\dot{f}$ is forced to U-uniformize $\vec{f}$. So $S \in U$. □(Claim)

Because $\mathbb{P}$ is $\leq \kappa$-closed, we can take $q \leq p$ and a sequence $\langle B_\xi \mid \xi \in S \rangle$ in $\mathcal{P}(\lambda)$ such that $q \Vdash$ "$\dot{f}(\xi) = B_\xi$" for all $\xi \in S$. Let $f : \kappa \to \mathcal{P}(\lambda)$ be such that $f(\xi) = B_\xi$ for all $\xi \in S$. From the choice of $\dot{f}$ and the claim above, it follows that f U-uniformizes $\vec{f}$. □

Now we prove Proposition 3.2:

Proof of Proposition 3.2. By Lemmata 3.1 and 3.4 it suffices to show that for any $\lambda \geq \mu$ every U-coherent sequence on $[\lambda]^{<\mu}$ is U-uniformizable. Suppose that $\lambda \geq \mu$ and that $\vec{f} = \langle f_x \mid x \in [\lambda]^{<\mu} \rangle$ is a U-coherent sequence. Let W be a $\leq \kappa$-closed forcing extension of V in which we have a (μ, λ)-strongly compact embedding $k : V \to N$. By Lemma 3.5 it suffices to show that $\vec{f}$ is U-uniformizable in W. We work in W.

Let $k(\vec{f}) = \langle g_y \mid y \in k([\lambda]^{<\mu}) \rangle$, and take $y^* \in k([\lambda]^{<\mu})$ such that $k[\lambda] \subseteq y^*$. Note that $g_y : \kappa \to \mathcal{P}(y)$ for each y. Now let $f : \kappa \to \mathcal{P}(\lambda)$ be the pull-back of g_{y^*} by k, that is,

$$f(\xi) = k^{-1}[\, g_{y^*}(\xi) \cap k[\lambda]\,]$$

for each $\xi < \kappa$. We claim that f U-uniformizes $\vec{f}$. Take an arbitrary $x \in [\lambda]^{<\mu}$. We must show that $\{\xi < \kappa \mid f(\xi) \cap x = f_x(\xi)\} \in U$. First note that $k[z] = k(z)$ for all $z \subseteq x$ because $|x| < \mu = \mathrm{crit}(k)$. Then for each $\xi < \kappa$,

$$\begin{aligned} f(\xi) \cap x = f_x(\xi) \quad &\Leftrightarrow \quad g_{y^*}(\xi) \cap k[x] = k[f_x(\xi)] \;\Leftrightarrow\; g_{y^*}(\xi) \cap k(x) = k(f_x(\xi)) \\ &\Leftrightarrow \quad g_{y^*}(\xi) \cap k(x) = g_{k(x)}(\xi)\ . \end{aligned}$$

Then

$$\{\xi < \kappa \mid f(\xi) \cap x = f_x(\xi)\} \;=\; \{\xi < \kappa \mid g_{y^*}(\xi) \cap k(x) = g_{k(x)}(\xi)\} \;\in\; k(U) \;=\; U\ ,$$

where the middle $\in$-relation is by the $k(U)$-coherency of $k(\vec{f}) = \langle g_y \mid y \in k([\lambda]^{<\mu}) \rangle$. □

3.3 Failure of μ-approximation property under $\Phi(\mu)$

Here we prove that M does not have the μ-approximation property under the following square-like principle $\Phi(\mu)$: For a regular cardinal $\mu > \kappa^+$ let

$\Phi(\mu) \equiv$ there are $E \subseteq \mathrm{Lim}(\mu)$ and $\langle c_\alpha \mid \alpha \in E \rangle$ such that

(i) $E^\mu_{>\kappa} \subseteq E$, and $E^\mu_\kappa \setminus E$ is stationary in μ,

(ii) c_α is a club subset of α for each $\alpha \in E$,

(iii) if $\alpha \in E$, and $\beta \in \mathrm{Lim}(c_\alpha)$, then $\beta \in E$, and $c_\alpha \cap \beta = c_\beta$.

First we observe that $\Phi(\nu^+)$ follows from Jensen's $\square_\nu$, which asserts the existence of a sequence $\langle c_\alpha \mid \alpha \in \mathrm{Lim}(\nu^+) \rangle$ such that

(i) each c_α is a club subset of α with $\mathrm{o.t.}(c_\alpha) \leq \nu$,

(ii) $c_\alpha \cap \beta = c_\beta$ if $\beta \in \mathrm{Lim}(c_\alpha)$.

Lemma 3.6. *Let ν be a cardinal $> \kappa$, and assume $\square_\nu$. Then $\Phi(\nu^+)$ holds.*

Proof. Let $\langle d_\alpha \mid \alpha \in \mathrm{Lim}(\nu^+) \rangle$ be a sequence witnessing $\square_\nu$. Then, because $\mathrm{o.t.}(d_\alpha) \leq \nu$ for all $\alpha \in E^{\nu^+}_\kappa$, there is $\rho \leq \nu$ such that $D := \{\alpha \in E^{\nu^+}_\kappa \mid \mathrm{o.t.}(d_\alpha) = \rho\}$ is stationary in ν^+. Let $E := \mathrm{Lim}(\nu^+) \setminus D$. For each $\alpha \in E$ define c_α as follows: If $\mathrm{o.t.}(d_\alpha) < \rho$, then let $c_\alpha := d_\alpha$. Otherwise, $\mathrm{o.t.}(d_\alpha) > \rho$. Let γ be the ρ-th element of d_α, and let $c_\alpha := d_\alpha \setminus \gamma$.

Now it is easy to check that E and $\langle c_\alpha \mid \alpha \in E \rangle$ witness $\Phi(\nu^+)$. $\square$

As we promised above, we prove the following:

Proposition 3.7. *Let μ be a regular cardinal $> \kappa^+$, and assume $\Phi(\mu)$. Then M does not have the μ-approximation property.*

Corollary 3.8. *Let ν be a cardinal $> \kappa$, and assume $\square_\nu$. Then M does not have the ν^+-approximation property.*

Proof of Proposition 3.7. Let E and $\langle c_\alpha \mid \alpha \in E \rangle$ be a pair witnessing $\Phi(\mu)$. By induction on $\alpha < \mu$ we will construct a U-coherent sequence $\langle f_\alpha \mid \alpha < \mu \rangle$ which is not U-uniformizable. The induction hypotheses are as follows:

(I) $[f_\alpha]_U \cap j(\beta) = [f_\beta]_U$ for each $\beta < \alpha$.

(II) If $\alpha \in E$, and $\beta \in \mathrm{Lim}(c_\alpha)$, then $f_\alpha(\xi) \cap \beta = f_\beta(\xi)$ for all $\xi < \kappa$.

Suppose that $\alpha < \mu$ and that $f_\beta : \kappa \to \mathcal{P}(\beta)$ has been taken for every $\beta < \alpha$.

Case 1: α is a successor ordinal.

Let $f_\alpha : \kappa \to \mathcal{P}(\alpha)$ be such that $[f_\alpha]_U = [f_{\alpha-1}]_U \cup \{j(\alpha - 1)\}$. Clearly f_α satisfies the induction hypotheses.

Case 2: $\alpha \in \mathrm{Lim}(\mu) \setminus E$.

In this case note that $\mathrm{cf}(\alpha) \leq \kappa$ by (i) of $\Phi(\mu)$. Let $B := \bigcup_{\beta<\alpha}[f_\beta]_U \subseteq j(\alpha)$. Then $B \in M$ because $\mathrm{cf}(\alpha) \leq \kappa$, and ${}^\kappa M \subseteq M$. Let $f_\alpha : \kappa \to \mathcal{P}(\alpha)$ be such that $[f_\alpha]_U = B$. Then f_α clearly satisfies the induction hypotheses. Here note that if $\mathrm{cf}(\alpha) = \kappa$, i.e. $\alpha \in E^\mu_\kappa \setminus E$, then $[f_\alpha]_U$ is bounded in $j(\alpha)$ because $B \subseteq \sup_{\beta<\alpha} j(\beta) < j(\alpha)$.

Case 3: $\alpha \in E$.

In this case note that if $\beta, \gamma \in \mathrm{Lim}(c_\alpha)$, and $\beta < \gamma$, then $\gamma \in E$ and $\beta \in \mathrm{Lim}(c_\gamma)$ by (iii) of $\Phi(\mu)$. So for such β, γ we have that $f_\gamma(\xi) \cap \beta = f_\beta(\xi)$ for all $\xi < \kappa$ by (II) for f_γ.

First suppose that $\mathrm{Lim}(c_\alpha)$ is unbounded in α. Define f_α by $f_\alpha(\xi) = \bigcup_{\gamma\in\mathrm{Lim}(c_\alpha)} f_\gamma(\xi)$ for all $\xi < \kappa$. Then f_α satisfies (II) by the remark above. Moreover it is easy to see that f_α also satisfies (I).

Next suppose that $\mathrm{Lim}(c_\alpha)$ is bounded in α. Let $\gamma := \max(\mathrm{Lim}(c_\alpha))$. Note that $\mathrm{cf}(\alpha) = \omega$ in this case. So we can take f_α satisfying (I) as in Case 2. Moreover we can take such f_α with the property that $f_\alpha(\xi) \cap \gamma = f_\gamma(\xi)$ for all $\xi < \kappa$. Then f_α also satisfies (II) by the remark above.

Now we have constructed a U-coherent $\vec{f} = \langle f_\alpha \mid \alpha < \mu \rangle$. By Lemma 3.1 it suffices to show that $\vec{f}$ is not U-uniformizable.

For the contradiction assume that $\vec{f}$ is U-uniformized by $f : \kappa \to \mathcal{P}(\mu)$. Note that $[f]_U \cap j(\alpha) = [f_\alpha]_U$ for all $\alpha < \mu$. Then $j[\mu] \subseteq [f]_U$ by the choice of f_α's for successor α's. Here note that $j[\mu]$ is unbounded in $j(\mu)$ because μ is a regular cardinal $> \kappa$. So $[f]_U$ is unbounded in $j(\mu)$, that is, $S := \{\xi < \kappa \mid f(\xi) \text{ is unbounded in } \mu\} \in U$. Then we can take $\alpha^* \in E^\mu_\kappa \setminus E$ such that $f(\xi) \cap \alpha^*$ is unbounded in α^* for all $\xi \in S$ because μ is a regular cardinal $> \kappa$, and $E^\mu_\kappa \setminus E$ is stationary in μ. Then $[f]_U \cap j(\alpha^*)$ is unbounded in $j(\alpha^*)$. But $[f]_U \cap j(\alpha^*) = [f_{\alpha^*}]_U$, and $[f_{\alpha^*}]_U$ is bounded in $j(\alpha^*)$ by the choice of f_{α^*} in Case 2. This is a contradiction. □

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